QUESTION NUMBER: 16

MODEL NUMBER: 1

TOKEN USAGE:

183 10936 11119 9792

TIME TAKEN: 95.7416 seconds

OUTPUT:

Given:

- OABC is a rectangle with vertices at O(0,0), A(8,0), B(8,4), and C(0,4)

- E is the midpoint of BC: E((8+0)/2, (4+4)/2) = (4,4)

- F is the midpoint of OA: F((0+8)/2, (0+0)/2) = (4,0)

- Point M(4,2) is the intersection point of AC and EF

- P(0, p) lies on the negative y-axis, so p < 0

- EF is the angle bisector of angle PED at E

- Circle c is centered at M and passes through D, the intersection of line PF and AC

- Circle c is tangent to both OA (y = 0) and BC (y = 4)

Our goal is to find the exact value of p for which circle c is tangent to both OA and BC.

We'll proceed step by step:

1. \*\*Find the equation of AC and the coordinates of D:\*\*

- Line AC passes through A(8,0) and C(0,4). Its equation is \( y = -0.5x + 4 \).

- Line PF passes through P(0,p) and F(4,0). Its equation is \( y = -\frac{p}{4}x + p \).

- Solving for the intersection D:

\[ -\frac{p}{4}x + p = -0.5x + 4 \]

\[ (-\frac{p}{4} + 0.5)x = p - 4 \]

\[ \left( \frac{-p+2}{4} \right) x = p - 4 \]

\[ x\_D = \frac{4(p - 4)}{-(p - 2)} = -\frac{4(p - 4)}{p - 2} \]

\[ y\_D = -0.5x\_D + 4 = \frac{2(p - 4)}{p - 2} + 4 \]

2. \*\*Use the angle bisector property at point E:\*\*

- The slopes of PE and DE are negative reciprocals because EF is the angle bisector:

\[ m\_{DE} = -m\_{PE} \]

- Compute \( m\_{PE} \):

\[ m\_{PE} = \frac{4 - p}{4} \]

- Compute \( m\_{DE} \):

\[ m\_{DE} = \frac{4 - y\_D}{4 - x\_D} \]

Substitute \( x\_D \) and \( y\_D \):

\[ m\_{DE} = -\frac{(p - 4)}{4(p - 3)} \]

- Set \( m\_{DE} = -m\_{PE} \):

\[ -\frac{(p - 4)}{4(p - 3)} = -\left( \frac{4 - p}{4} \right) \]

- Solving this equation, we get:

\[ \frac{p - 4}{4(p - 3)} = \frac{p - 4}{4} \]

\[ \frac{1}{p - 3} = 1 \]

\[ p - 3 = 1 \]

\[ p = 4 \]

However, \( p = 4 \) is invalid since P is on the negative y-axis.

3. \*\*Compute the radius of circle c centered at M:\*\*

- Since circle c is tangent to OA and BC (y = 0 and y = 4) and centered at \( M(4,2) \), its radius is 2.

- The distance from M to D must be equal to 2:

\[ (x\_D - 4)^2 + (y\_D - 2)^2 = 4 \]

Substitute \( x\_D \) and \( y\_D \) and simplify:

\[ (-\frac{8p - 24}{p - 2})^2 + (\frac{4p - 12}{p - 2})^2 = 4 \]

Simplify and solve the resulting quadratic equation:

\[ 19p^2 - 116p + 176 = 0 \]

Discriminant:

\[ D = (-116)^2 - 4 \cdot 19 \cdot 176 = 80 \]

Roots:

\[ p = \frac{116 \pm 4\sqrt{5}}{38} \]

Simplifying, one of the roots is negative:

\[ p = \frac{-58 + 2\sqrt{5}}{19} \]

- As p must be negative, the acceptable value is:

\[ p = \frac{-58 + 2\sqrt{5}}{19} \]

\*\*Answer:\*\*

An explicit value: p = [–58 + 2 √5] ⁄ 19

QUESTION NUMBER: 16

MODEL NUMBER: 1

TOKEN USAGE:

183 10474 10657 9728

TIME TAKEN: 90.4327 seconds

OUTPUT:

The problem is about determining the specific value of \( p \) such that a circle centered at \( M(4, 2) \) and passing through point \( D \) is tangent to both \( OA \) and \( BC \) in the given rectangle \( OABC \).

First, recall that the circle will be tangent to \( OA \) and \( BC \) when its radius is equal to the vertical distance from \( M \) to either \( OA \) or \( BC \), which is \( 2 \) units. Therefore, the circle's radius \( r \) should satisfy \( r = 2 \).

We need to find \( p \) so that the distance \( MD \) is \( 2 \). We find \( D \) by solving for the intersection of lines \( PF \) and \( AC \):

1. Equation of \( PF \):

\[

m\_{PF} = \frac{0 - p}{4 - 0} = -\frac{p}{4}

\]

\[

y = -\frac{p}{4}x + p

\]

2. Equation of \( AC \):

\[

m\_{AC} = \frac{4 - 0}{0 - 8} = -\frac{1}{2}

\]

\[

y = -\frac{1}{2}x + 4

\]

3. Solving for \( x\_D \) and \( y\_D \) at the intersection:

\[

-\frac{p}{4}x\_D + p = -\frac{1}{2}x\_D + 4

\]

Solving for \( x\_D \) and \( y\_D \):

\[

x\_D = \frac{16 - 4p}{p + 2}, \quad y\_D = -\frac{1}{2}x\_D + 4

\]

Now, calculate the distance \( MD \):

\[

MD^2 = (x\_D - 4)^2 + (y\_D - 2)^2 = 4

\]

Substituting \( x\_D \) and \( y\_D \) and simplifying, we get:

\[

80(1 - p)^2 = 4(p + 2)^2

\]

\[

20(1 - p)^2 = (p + 2)^2

\]

Solving this quadratic equation, we obtain two values of \( p \), but only one is applicable (since \( p \) must be on the negative y-axis and match the geometric constraints). After solving, we find:

\[

p = \frac{22 - 6\sqrt{5}}{19}

\]

Therefore, the exact value of \( p \) where the circle \( c \) is tangent to both \( OA \) and \( BC \) is:

\[

p = \frac{22 - 6\sqrt{5}}{19}

\]

\*\*Answer:\*\* An exact value of p = (22 minus 6 × √5) divided by 19; that is,

 p = [22 – 6√5] ⁄ 19.

QUESTION NUMBER: 16

MODEL NUMBER: 1

TOKEN USAGE:

183 5134 5317 4288

TIME TAKEN: 45.7836 seconds

OUTPUT:

To solve this problem, we'll first recreate the essential elements:

\*\*Step 1: Determine Important Points and Lines\*\*

- The rectangle OABC has vertices at O(0, 0), A(8, 0), B(8, 4), and C(0, 4).

- Midpoints:

- \( F \) is the midpoint of \( OA \), so \( F(4, 0) \).

- \( E \) is the midpoint of \( BC \), so \( E(4, 4) \).

- Line \( AC \): It passes through A(8, 0) and C(0, 4). Its equation is \( y = -0.5x + 4 \).

- Line \( PF \): It connects P(0, p) and F(4, 0). Its equation is \( y = \frac{-p}{4}(x - 4) \).

\*\*Step 2: Find \( D \), the Intersection of \( PF \) and \( AC \)\*\*

Set \( y \) from \( PF \) equal to \( y \) from \( AC \):

\[

\frac{-p}{4}(x - 4) = -0.5x + 4.

\]

Simplify and solve for \( x\_D \):

\[

-p(x - 4) = -2x + 16 \\

-px + 4p + 2x - 16 = 0 \\

x(2 - p) = 16 - 4p \\

x\_D = \frac{16 - 4p}{2 - p}.

\]

Find \( y\_D \):

\[

y\_D = -0.5x\_D + 4.

\]

\*\*Step 3: Compute \( MD^2 \) Using the Circle Center \( M(4, 2) \)\*\*

Calculate \( x\_D - 4 \) and \( y\_D - 2 \):

\[

x\_D - 4 = \frac{8}{2 - p}, \\

y\_D - 2 = \frac{-4}{2 - p}.

\]

Compute \( MD^2 \):

\[

MD^2 = (x\_D - 4)^2 + (y\_D - 2)^2 \\

= \left( \frac{8}{2 - p} \right)^2 + \left( \frac{-4}{2 - p} \right)^2 \\

= \frac{64 + 16}{(2 - p)^2} \\

= \frac{80}{(2 - p)^2}.

\]

\*\*Step 4: Set \( MD^2 \) Equal to 4 (Radius of Circle c Touching OA and BC)\*\*

Since the circle must be tangent to both \( y = 0 \) (OA) and \( y = 4 \) (BC), its radius is 2:

\[

\frac{80}{(2 - p)^2} = 4.

\]

Simplify:

\[

(2 - p)^2 = \frac{80}{4} \\

(2 - p)^2 = 20 \\

2 - p = \pm 2\sqrt{5} \\

p = 2 \mp 2\sqrt{5}.

\]

\*\*Step 5: Select the Appropriate Value of \( p \)\*\*

Given that P is on the negative y-axis ( \( p < 0 \) ), we select:

\[

p = 2 - 2\sqrt{5}.

\]

Therefore, the exact value of \( p \) is \( 2 - 2\sqrt{5} \).

\*\*Answer: p = 2 minus 2 times √5; that is, p = 2 – 2·√5\*\*

QUESTION NUMBER: 16

MODEL NUMBER: 1

TOKEN USAGE:

183 8359 8542 7232

TIME TAKEN: 75.0713 seconds

OUTPUT:

To solve this problem, we start by analyzing the given geometric configuration:

1. \*\*Rectangle OABC\*\*: O(0,0), A(8,0), B(8,4), and C(0,4).

2. \*\*Points F and E\*\*: Midpoints of sides OA and BC, respectively.

- F is the midpoint of OA, so F(4,0).

- E is the midpoint of BC, so E(4,4).

3. \*\*Point P\*\*: Lies on the negative y-axis, so P(0, p) with p < 0.

4. \*\*Line PF\*\*: Connects P and F.

5. \*\*Point D\*\*: Intersection of line PF extended and side AC.

6. \*\*Point M\*\*: Intersection of line EF (which is the angle bisector of angle PED) and AC. Given that M(4,2).

7. \*\*Circle c\*\*: Centered at M and passing through D.

Our goal is to find the exact value of p such that circle c is tangent to sides OA and BC.

\*\*Step-by-Step Solution:\*\*

\*\*A. Find the coordinates of D in terms of p:\*\*

- \*\*Equation of line PF:\*\*

- Slope, \( m\_{PF} = \frac{0 - p}{4 - 0} = \frac{-p}{4} \)

- Equation: \( y - p = \frac{-p}{4} \cdot x \)

- \*\*Equation of line AC:\*\*

- A(8,0) and C(0,4) yield the equation \( y = -\frac{1}{2}x + 4 \)

- \*\*Find intersection point D(x₁, y₁):\*\*

- Set \( y \) from PF equal to \( y \) from AC:

\[

\frac{-p}{4}x + p = -\frac{1}{2}x + 4

\]

- Solve for x:

\[

x = \frac{4 - p}{\frac{-p}{4} + \frac{1}{2}} \cdot 4 = \frac{4(4 - p)}{2 - p}

\]

- Find \( y \) using AC:

\[

y = -\frac{1}{2}x + 4 = -\frac{1}{2} \cdot \frac{4(4 - p)}{2 - p} + 4 = \frac{-2(4 - p)}{2 - p} + 4

\]

\*\*B. Calculate the radius of circle c:\*\*

- The radius \( r \) is the distance from M(4,2) to D.

- Compute \( MD^2 \):

\[

MD^2 = (x\_D - 4)^2 + (y\_D - 2)^2

\]

- \( x\_D - 4 = \frac{4(4 - p)}{2 - p} - 4 = \frac{8}{2 - p} \)

- \( y\_D - 2 = \left( \frac{-2(4 - p)}{2 - p} + 4 \right) - 2 = \frac{-4}{2 - p} \)

- Compute \( MD^2 \):

\[

MD^2 = \left( \frac{8}{2 - p} \right)^2 + \left( \frac{-4}{2 - p} \right)^2 = \frac{64 + 16}{(2 - p)^2} = \frac{80}{(2 - p)^2}

\]

- \( r^2 = \frac{80}{(2 - p)^2} \)

\*\*C. Set up the condition for the circle to be tangent to OA and BC:\*\*

- The circle is tangent to OA (y = 0) and BC (x = 8).

- Distance from M to OA is \( |2 - 0| = 2 \).

- Distance from M to BC is \( |4 - 8| = 4 \).

- Thus, for the circle to be tangent to both, the radius must satisfy \( r = 2 \) and \( r = 4 \). However, since both conditions must be satisfied simultaneously, we need to use the largest distance (4) to ensure tangency to both sides.

- Therefore, set \( r^2 = 4^2 = 16 \).

\*\*D. Solve for p:\*\*

- Equate the expressions for \( r^2 \):

\[

\frac{80}{(2 - p)^2} = 16

\]

- Solve for \( p \):

\[

(2 - p)^2 = \frac{80}{16} = 5 \\

2 - p = \pm \sqrt{5} \\

p = 2 \mp \sqrt{5}

\]

- Since p < 0, choose \( p = 2 - \sqrt{5} \).

\*\*Answer:\*\*

\( p = 2 - \sqrt{5} \)

QUESTION NUMBER: 16

MODEL NUMBER: 1

TOKEN USAGE:

183 7530 7713 6784

TIME TAKEN: 68.7185 seconds

OUTPUT:

We are given a rectangle OABC with O(0,0), A(8,0), and C(0,4). Points F and E are midpoints of OA and BC respectively, so F(4,0) and E(4,4). The line EF is vertical at x=4, and intersects AC at M(4,2). Point P is at (0,p) on the negative y-axis.

Our goal is to find the exact value of \( p \) such that the circle centered at M(4,2) passing through point D is tangent to OA and BC.

First, we find coordinates of point D. Line AC has the equation \( y = -\frac{1}{2}x + 4 \). Line PF passes through P(0,p) and F(4,0) with equation \( y = -\frac{p}{4}x + p \). Setting these equal to find their intersection (point D), we get:

\[

-\frac{p}{4}x + p = -\frac{1}{2}x + 4 \implies x\_D = \frac{16 - 4p}{2 - p}

\]

\[

y\_D = -\frac{2p}{2 - p}

\]

Next, since EF is the angle bisector of angle PED at E(4,4), we find the slopes of EP and ED:

\[

m\_{EP} = \frac{4 - p}{4}, \quad m\_{ED} = \frac{p - 4}{4}

\]

The angle bisector condition gives \( m\_{EP} = -m\_{ED} \), which is always true here.

Now, for the circle centered at M to be tangent to OA and BC (which are at y=0 and y=4), the radius \( r \) must be equal to 2 (the distance from M to OA or BC). So, we set the distance from M to D equal to 2:

\[

MD^2 = \left( x\_D - 4 \right)^2 + \left( y\_D - 2 \right)^2 = 4

\]

Computing \( x\_D - 4 \) and \( y\_D - 2 \):

\[

x\_D - 4 = \frac{8}{2 - p}, \quad y\_D - 2 = \frac{-4}{2 - p}

\]

So:

\[

MD^2 = \left( \frac{8}{2 - p} \right)^2 + \left( \frac{-4}{2 - p} \right)^2 = \frac{80}{(2 - p)^2}

\]

Setting this equal to 4:

\[

\frac{80}{(2 - p)^2} = 4 \implies (2 - p)^2 = 20 \implies 2 - p = \pm 2\sqrt{5}

\]

Since P is on the negative y-axis (p < 0), we take the positive root:

\[

2 - p = 2\sqrt{5} \implies p = 2 - 2\sqrt{5}

\]

\*\*Answer:\*\* An explicit value: p = 2 minus 2 times √5; that is, p = 2 – 2·√5.

QUESTION NUMBER: 16

MODEL NUMBER: 1

TOKEN USAGE:

183 7918 8101 6784

TIME TAKEN: 74.1957 seconds

OUTPUT:

To solve this problem, we'll work through several geometric and algebraic steps.

\*\*1. Define Given Points and Lines:\*\*

- \*\*Rectangle OABC\*\* with vertices:

- \(O(0,0)\)

- \(A(8,0)\)

- \(B(8,4)\)

- \(C(0,4)\)

- \*\*Midpoints:\*\*

- \(F\) is the midpoint of \(OA\): \(F\left(\frac{0+8}{2},\frac{0+0}{2}\right) = (4,0)\)

- \(E\) is the midpoint of \(BC\): \(E\left(\frac{8+0}{2},\frac{4+4}{2}\right) = (4,4)\)

- \*\*Line \(EF\):\*\* Vertical line \(x = 4\).

- \*\*Point \(P\):\*\* \((0, p)\) on the negative \(y\)-axis, so \(p < 0\).

- \*\*Line \(PF\):\*\* Connects \(P\) and \(F\).

- \*\*Diagonal \(AC\):\*\* Connects \(A(8,0)\) and \(C(0,4)\) with equation \(y = -0.5x + 4\).

- \*\*Intersection Point \(M\):\*\* Intersection of \(EF\) and \(AC\): \(M(4,2)\).

\*\*2. Find Point \(D\):\*\*

\(D\) is the intersection of the extension of \(PF\) and \(AC\).

- \*\*Equation of \(PF\):\*\*

\[

\text{Slope } m\_{PF} = \frac{0 - p}{4 - 0} = \frac{-p}{4}

\]

\[

y - p = m\_{PF} (x - 0)

\]

- \*\*Equation of \(AC\):\*\*

\[

y = -0.5x + 4

\]

- \*\*Find \(x\_D\) and \(y\_D\):\*\* Solve these two equations simultaneously.

\*\*3. Use the Angle Bisector Property:\*\*

- Since \(EF\) bisects angle \(PED\) at \(E\), and \(EF\) is vertical (\(x=4\)), the slopes of \(EP\) and \(ED\) satisfy:

\[

m\_{ED} = -m\_{EP}

\]

- \*\*Calculate Slopes:\*\*

\[

m\_{EP} = \frac{p - 4}{0 - 4} = \frac{4 - p}{4}

\]

\[

m\_{ED} = \frac{p - 4}{4} = -m\_{EP}

\]

\*\*4. Find the Coordinates of \(D\):\*\*

- Using the slope \(m\_{ED}\) and point \(E(4,4)\):

\[

y - 4 = m\_{ED}(x - 4)

\]

- Combine with the equation of \(AC\) to solve for \(x\_D\) and \(y\_D\):

\[

x\_D = \frac{16 - 4p}{2 - p}, \quad y\_D = \frac{-2p}{2 - p}

\]

\*\*5. Compute the Radius of Circle \(c\):\*\*

- Radius \(r\) is the distance from \(M(4,2)\) to \(D(x\_D, y\_D)\):

\[

MD^2 = (x\_D - 4)^2 + (y\_D - 2)^2

\]

- Simplify \(MD^2\):

\[

MD^2 = \frac{80}{(2 - p)^2}

\]

\*\*6. Set the Circle to be Tangent to \(OA\) and \(BC\):\*\*

- For circle \(c\) to be tangent to both \(y=0\) and \(y=4\), its radius must be \(r = 2\) because the center \(M\) is at \(y=2\), so it's \(2\) units away from both lines.

- Set \(MD^2 = r^2\):

\[

\frac{80}{(2 - p)^2} = 4

\]

- \*\*Solve for \(p\):\*\*

\[

\frac{80}{(2 - p)^2} = 4 \implies 80 = 4(2 - p)^2 \implies (2 - p)^2 = 20 \implies 2 - p = \pm 2\sqrt{5}

\]

- Since \(p < 0\), choose the solution:

\[

2 - p = 2\sqrt{5} \implies p = 2 - 2\sqrt{5}

\]

\*\*Answer:\*\*

An exact value of p = 2 minus 2 times √5; that is, p = 2 – 2√5

QUESTION NUMBER: 16

MODEL NUMBER: 1

TOKEN USAGE:

183 3761 3944 3008

TIME TAKEN: 34.1271 seconds

OUTPUT:

To solve this problem, we need to find the exact value of \( p \) such that circle \( c \), centered at \( M(4,2) \) and passing through point \( D \), is tangent to both lines \( OA \) and \( BC \). Here's how we can approach this step by step:

1. \*\*Find Equations of Lines and Points:\*\*

- \*\*Line \( PF \):\*\* Passes through \( P(0,p) \) and \( F(4,0) \).

\[ \text{Slope of } PF = \frac{0 - p}{4 - 0} = -\frac{p}{4} \]

\[ \text{Equation: } y - p = -\frac{p}{4} x \]

- \*\*Line \( AC \):\*\* Passes through \( A(8,0) \) and \( C(0,4) \).

\[ \text{Slope of } AC = \frac{4 - 0}{0 - 8} = -\frac{1}{2} \]

\[ \text{Equation: } y = -\frac{1}{2}(x - 8) \]

2. \*\*Find Point \( D \):\*\* Intersection of \( PF \) and \( AC \).

- Equate \( y \) from both lines:

\[ -\frac{p}{4} x + p = -\frac{1}{2} x + 4 \]

\[ \Rightarrow \left(-\frac{p}{4} + \frac{1}{2}\right) x = p - 4 \]

\[ \Rightarrow x = \frac{4(p - 4)}{p - 2} \]

- Substitute \( x \) back to find \( y \):

\[ y = -\frac{1}{2} x + 4 \]

\[ \Rightarrow y = \frac{2p}{p - 2} \]

3. \*\*Compute \( (x - 4)^2 + (y - 2)^2 \):\*\* Since circle \( c \) has center \( M(4,2) \) and radius \( r \).

- The circle must be tangent to \( y = 0 \) and \( y = 4 \), so \( r = 2 \).

- Calculate:

\[ (x - 4)^2 + (y - 2)^2 = r^2 \]

\[ \left(\frac{-8}{p - 2}\right)^2 + \left(\frac{4}{p - 2}\right)^2 = 4 \]

\[ \frac{64 + 16}{(p - 2)^2} = 4 \]

\[ \frac{80}{(p - 2)^2} = 4 \]

\[ (p - 2)^2 = 20 \]

\[ p - 2 = \pm 2\sqrt{5} \]

4. \*\*Determine the Valid Value of \( p \):\*\*

- Since \( P \) lies on the negative \( y \)-axis (\( p < 0 \)):

\[ p = 2 - 2\sqrt{5} \]

\*\*Answer:\*\*

\[ p = 2 - 2\sqrt{5} \]

QUESTION NUMBER: 16

MODEL NUMBER: 1

TOKEN USAGE:

183 6049 6232 5248

TIME TAKEN: 49.2886 seconds

OUTPUT:

To solve this problem, we start by analyzing the given rectangle \( OABC \) with \( O(0,0) \), \( A(8,0) \), and \( C(0,4) \). Points \( F \) and \( E \) are the midpoints of \( OA \) and \( BC \), located at \( F(4,0) \) and \( E(4,4) \) respectively. The line \( EF \) is vertical at \( x = 4 \).

Point \( P(0, p) \) lies on the negative \( y \)-axis, so \( p < 0 \). Line \( PF \) connects \( P \) and \( F \), and its equation is:

\[ y - p = \left( \frac{-p}{4} \right)(x - 0) \implies y = p - \left( \frac{p}{4} \right)x \]

Line \( AC \) runs from \( A(8,0) \) to \( C(0,4) \) and has the equation:

\[ y = -\frac{1}{2}x + 4 \]

Point \( D \) is the intersection of lines \( PF \) and \( AC \). Setting \( y \) from both equations equal:

\[ p - \left( \frac{p}{4} \right)x = -\frac{1}{2}x + 4 \]

Solving for \( x \) gives:

\[ x\_D = \frac{16 - 4p}{2 - p} \]

Given that \( EF \) bisects angle \( PED \) at \( E(4,4) \), and \( EF \) is vertical, the reflection of line \( EP \) over \( x = 4 \) coincides with line \( ED \). The reflection of \( P(0,p) \) over \( x = 4 \) is \( P'(8,p) \). Thus, \( D \) lies on both \( AC \) and line \( EP' \), whose slope is:

\[ m\_{EP'} = \frac{p - 4}{8 - 4} = \frac{p - 4}{4} \]

Using the equation of \( AC \) and the equation of \( ED \), we find that \( x\_D \) matches our previous expression.

Circle \( c \) centered at \( M(4,2) \) passes through \( D \). For circle \( c \) to be tangent to \( OA \) (\( y = 0 \)) and \( BC \) (\( y = 4 \)), its radius must be \( 2 \) units. Therefore, we set the distance \( MD \) equal to \( 2 \) and solve:

\[

MD^2 = (x\_D - 4)^2 + (y\_D - 2)^2 = 4

\]

Substituting \( x\_D \) and \( y\_D \) into the equation and simplifying, we find:

\[

-16s^2 + 320 = 0 \quad \text{where} \quad s = 2 - p

\]

Solving for \( s \) gives \( s = 2\sqrt{5} \). Therefore,

\[

p = 2 - s = 2 - 2\sqrt{5}

\]

\*\*Answer:\*\* An explicit value: p = 2 minus 2 times √5; that is, p = 2 – 2√5

QUESTION NUMBER: 16

MODEL NUMBER: 1

TOKEN USAGE:

183 6275 6458 5440

TIME TAKEN: 60.8511 seconds

OUTPUT:

To solve this problem, we'll work step by step through the given scenario.

\*\*Given:\*\*

- Rectangle OABC with vertices at O(0,0), A(8,0), C(0,4), and by implication, B(8,4).

- F is the midpoint of OA, so F(4,0).

- E is the midpoint of BC, so E(4,4).

- M(4,2) is the intersection point of AC and EF.

- Point P(0,p) lies on the negative y-axis (so p < 0).

\*\*Objective:\*\*

Find the value of \( p \) such that circle \( c \) with center \( M(4,2) \) and passing through \( D \) is tangent to both lines OA and BC.

\*\*Steps:\*\*

1. \*\*Find the Equation of Line PF:\*\*

The line PF passes through points \( P(0,p) \) and \( F(4,0) \).

The slope is \( m\_{PF} = \frac{0 - p}{4 - 0} = -\frac{p}{4} \).

The equation is \( y = -\frac{p}{4} x + p \).

2. \*\*Find the Intersection Point D of Lines PF and AC:\*\*

The line AC goes from \( A(8,0) \) to \( C(0,4) \) and has the equation \( y = -\frac{1}{2} x + 4 \).

Set the equations of PF and AC equal to find \( x\_D \):

\[

-\frac{p}{4} x + p = -\frac{1}{2} x + 4 \\

\]

Solving for \( x\_D \) gives:

\[

x\_D = \frac{16 - 4p}{-p + 2}

\]

Then, \( y\_D = -\frac{1}{2} x\_D + 4 \).

3. \*\*Compute \( x\_D - 4 \) and \( y\_D - 2 \):\*\*

\[

x\_D - 4 = \frac{8}{-p + 2} \\

y\_D - 2 = -\frac{4}{-p + 2}

\]

4. \*\*Compute the Square of the Distance \( MD^2 \):\*\*

\[

MD^2 = (x\_D - 4)^2 + (y\_D - 2)^2 = \frac{80}{(-p + 2)^2}

\]

5. \*\*Set the Radius Equal to the Distance from M to OA:\*\*

The circle is tangent to OA, so the radius \( r \) equals the vertical distance from \( M \) to OA, which is 2 units.

So, \( MD^2 = 2^2 = 4 \).

\[

\frac{80}{(-p + 2)^2} = 4 \\

(-p + 2)^2 = 20 \\

-p + 2 = \pm 2\sqrt{5} \\

\]

Since \( p < 0 \), we take:

\[

-p + 2 = 2\sqrt{5} \\

-p = 2\sqrt{5} - 2 \\

p = 2 - 2\sqrt{5}

\]

Thus, the exact value of \( p \) is \( 2 - 2\sqrt{5} \).

\*\*Answer: p = 2 minus 2 times √5; that is, p = 2 – 2√5\*\*

QUESTION NUMBER: 16

MODEL NUMBER: 1

TOKEN USAGE:

183 4724 4907 4096

TIME TAKEN: 45.608 seconds

OUTPUT:

To solve this problem, we need to find the value of \( p \) such that a circle centered at \( M(4, 2) \) passing through \( D \) is tangent to both \( OA \) and \( BC \) of rectangle \( OABC \).

Here are the steps:

1. \*\*Determine Coordinates of \( D \):\*\*

- The line \( PF \) passes through \( P(0, p) \) and \( F(4, 0) \).

- The equation of \( PF \) is \( y = \left( -\frac{p}{4} \right) x + p \).

- The line \( AC \) has the equation \( y = -\frac{1}{2} x + 4 \).

- Solving \( PF \) and \( AC \) gives the intersection point \( D \).

2. \*\*Express Coordinates of \( D \) in Terms of \( p \):\*\*

- After solving the equations, \( D \) has coordinates:

\[

x = \frac{16 - 4p}{-p + 2}, \quad y = \frac{2p}{p - 2}

\]

3. \*\*Find Radius \( r \) of Circle \( c \):\*\*

- The distance from \( M(4, 2) \) to \( D \) is:

\[

r^2 = (x - 4)^2 + (y - 2)^2 = \frac{80}{(-p + 2)^2}

\]

4. \*\*Circle Tangency Conditions:\*\*

- For the circle to be tangent to \( OA \) (which lies along \( y = 0 \)), the radius must equal the \( y \)-distance from \( M \) to \( y = 0 \): \( r = 2 \).

- For the circle to be tangent to \( BC \) (which lies along \( x = 8 \)), the radius must equal the \( x \)-distance from \( M \) to \( x = 8 \): \( r = 4 \).

- Since the circle must satisfy both conditions simultaneously, we set \( r = 4 \).

5. \*\*Solve for \( p \):\*\*

- Equate the expressions for \( r^2 \) and solve for \( p \):

\[

16 = \frac{80}{(-p + 2)^2} \implies (-p + 2)^2 = 5 \implies p = 2 \pm \sqrt{5}

\]

- Since \( P \) lies on the negative \( y \)-axis, \( p \) must be negative.

- Therefore, \( p = 2 - \sqrt{5} \).

\*\*Answer:\*\* \(p = 2 - \sqrt{5}\)

QUESTION NUMBER: 16

MODEL NUMBER: 1

TOKEN USAGE:

183 9378 9561 8576

TIME TAKEN: 83.9742 seconds

OUTPUT:

To solve the problem, we first analyze the given geometric figure and relationships. We are given a rectangle OABC with vertices at O(0, 0), A(8, 0), B(8, 4), and C(0, 4). Points F(4, 0) and E(4, 4) are the midpoints of OA and BC, respectively. Point P is at (0, p) on the negative y-axis (so \(p < 0\)), and M(4, 2) is given as the intersection point of line AC and the line EF, which is the line \(x = 4\).

The circle \(c\) has center M(4, 2) and passes through point D, which is the point of intersection of line PF and line AC. We are to find the value of \(p\) such that circle \(c\) is tangent to lines OA (which lies along \(y = 0\)) and BC (which lies along \(x = 8\)).

Steps:

1. \*\*Find the coordinates of D in terms of \(p\):\*\*

- The equation of line PF is \(y = -\frac{p}{4} x + p\).

- The equation of line AC is \(y = -\frac{1}{2} x + 4\).

- Solving these equations simultaneously gives \(x\_D = \frac{16 - 4p}{-p + 2}, y\_D = -\frac{1}{2} x\_D + 4\).

2. \*\*Use the angle bisector property:\*\*

- The line \(x = 4\) is the angle bisector of angle PED at E(4, 4). This implies that slopes \(m\_{EP}\) and \(m\_{ED}\) satisfy \(m\_{EP} = -m\_{ED}\).

- Calculate \(m\_{EP} = \frac{4 - p}{4}\) and \(m\_{ED} = \frac{-4 + p}{4}\), confirming that \(m\_{EP} + m\_{ED} = 0\).

3. \*\*Set up the circle equations:\*\*

- Since the circle must be tangent to \(y = 0\) and \(x = 8\), the center \((h, k)\) satisfies \(k = r\) and \(h = 8 - r\).

- Using the point M(4, 2) on the circle, we have \((4 - h)^2 + (2 - k)^2 = r^2\).

4. \*\*Solve for \(r\):\*\*

- Substitute \(h = 8 - r\) and \(k = r\) into the circle equation, leading to \(r = 2\).

5. \*\*Find the center \((h, k)\) and confirm tangency:\*\*

- With \(r = 2\), \(h = 6\) and \(k = 2\).

- The circle centered at (6, 2) with radius 2 passes through M(4, 2) and is tangent to both \(y = 0\) and \(x = 8\).

6. \*\*Find \(p\) using the circle equation:\*\*

- Substitute \((x\_D, y\_D)\) into the circle equation \((x\_D - h)^2 + (y\_D - k)^2 = r^2\).

- Simplify and solve the resulting equation, which yields \(p = -\frac{1}{2}\).

\*\*Answer:\*\*

\(\boxed{\displaystyle -\frac{1}{2}}\)

QUESTION NUMBER: 16

MODEL NUMBER: 1

TOKEN USAGE:

183 9057 9240 8448

TIME TAKEN: 77.6924 seconds

OUTPUT:

We need to find the exact value of \( p \) such that circle \( c \), centered at \( M(4,2) \) and passing through point \( D \), is tangent to both line segments \( OA \) and \( BC \) of rectangle \( OABC \).

Firstly, we note that for the circle to be tangent to \( OA \) (the \( x \)-axis), the vertical distance from \( M \) to \( OA \) must be equal to the radius \( r \) of the circle. Similarly, for the circle to be tangent to \( BC \) (the vertical line \( x = 8 \)), the horizontal distance from \( M \) to \( BC \) must also be equal to the radius \( r \).

So, the distances are:

- Vertical distance from \( M \) to \( OA \): \( |2 - 0| = 2 \)

- Horizontal distance from \( M \) to \( BC \): \( |4 - 8| = 4 \)

Thus, for the circle to be tangent to both \( OA \) and \( BC \), the radius must simultaneously satisfy \( r = 2 \) and \( r = 4 \), which is impossible. But since the circle is centered at \( M \) and passes through \( D \), we consider the radius dependent on \( p \) as \( r^2 = \frac{80}{(2 - p)^2} \).

To find \( p \) such that the circle is tangent to \( OA \), set \( r^2 = 4 \):

\[

\frac{80}{(2 - p)^2} = 4 \implies (2 - p)^2 = 20 \implies 2 - p = \pm 2\sqrt{5}

\]

Since \( p \) is negative (point \( P \) is on the negative \( y \)-axis), we take the solution:

\[

p = 2 - 2\sqrt{5}

\]

Similarly, to find \( p \) such that the circle is tangent to \( BC \), set \( r^2 = 16 \):

\[

\frac{80}{(2 - p)^2} = 16 \implies (2 - p)^2 = 5 \implies 2 - p = \pm \sqrt{5}

\]

Again, since \( p \) is negative, we take:

\[

p = 2 - \sqrt{5}

\]

The only value of \( p \) that satisfies both conditions (as an exact value on the negative \( y \)-axis) is:

\[

p = 2 - \sqrt{5}

\]

\*\*Answer:\*\* \(p = 2\ -\ \sqrt{5}\)

QUESTION NUMBER: 16

MODEL NUMBER: 1

TOKEN USAGE:

183 4555 4738 3584

TIME TAKEN: 37.4239 seconds

OUTPUT:

Let’s work through the problem step by step.

\*\*Coordinates and Important Points:\*\*

1. \*\*Rectangle\*\* OABC with vertices O(0,0), A(8,0), C(0,4). Hence, B must be at (8,4).

2. \*\*Midpoints:\*\*

- \*\*F\*\*, the midpoint of OA:

\[ F = \left( \frac{0+8}{2}, \frac{0+0}{2} \right) = (4,0) \]

- \*\*E\*\*, the midpoint of BC:

\[ E = \left( \frac{8+0}{2}, \frac{4+4}{2} \right) = (4,4) \]

- So the line EF is vertical at \( x = 4 \).

3. \*\*Point P\*\* lies on the negative y-axis at \( (0,p) \).

4. \*\*Line PF\*\* has a slope of:

\[ m\_{PF} = \frac{0 - p}{4 - 0} = \frac{-p}{4} \]

Thus, the equation of PF is:

\[ y = \frac{-p}{4} x + p \]

5. \*\*Line AC\*\* passes through A(8,0) and C(0,4). Its slope is:

\[ m\_{AC} = \frac{4 - 0}{0 - 8} = -\frac{1}{2} \]

The equation is:

\[ y = -\frac{1}{2} x + 4 \]

6. \*\*Point D\*\* is the intersection of PF and AC:

- Equate y-values:

\[ \frac{-p}{4} x + p = -\frac{1}{2} x + 4 \]

- Solve for \( x \):

\[ \left( \frac{-p}{4} + \frac{1}{2} \right) x = 4 - p \]

\[ \left( \frac{2 - p}{4} \right) x = 4 - p \]

\[ x = \frac{4(4 - p)}{2 - p} \]

- Compute \( y \):

\[ y = -\frac{1}{2} x + 4 = -\frac{1}{2} \left( \frac{4(4 - p)}{2 - p} \right) + 4 \]

Simplify to find \( y \) in terms of \( p \).

7. \*\*Coordinates of D\*\*:

- After simplification, \( x\_D = \frac{-16 + 4p}{p - 2} \), \( y\_D = \frac{2p}{p - 2} \).

8. \*\*Distance from M(4,2) to D\*\*:

- The distance squared is:

\[ (x\_D - 4)^2 + (y\_D - 2)^2 = r^2 \]

9. \*\*Condition for tangency\*\*:

- Since circle \( c \) is tangent to both OA (at \( y = 0 \)) and BC (at \( y = 4 \)), and centered at M(4,2), its radius must be:

\[ r = 2 \]

10. \*\*Set up the equation\*\*:

\[ \left( \frac{-8}{p - 2} \right)^2 + \left( \frac{4}{p - 2} \right)^2 = 4 \]

Simplify:

\[ \frac{64 + 16}{(p - 2)^2} = 4 \]

\[ \frac{80}{(p - 2)^2} = 4 \]

\[ (p - 2)^2 = 20 \]

\[ p - 2 = \pm 2\sqrt{5} \]

11. \*\*Since P lies on the negative y-axis\*\*, \( p \) must be negative:

\[ p = 2 - 2\sqrt{5} \]

\*\*Answer:\*\*

\[ p = 2 - 2\sqrt{5} \]

Thus, the exact value of \( p \) is \( 2 - 2\sqrt{5} \).

\*\*Answer:\*\* \(p = 2 - 2\sqrt{5}\)

QUESTION NUMBER: 16

MODEL NUMBER: 1

TOKEN USAGE:

183 8837 9020 7808

TIME TAKEN: 86.0635 seconds

OUTPUT:

To solve the problem, we need to find the exact value of \( p \) for which circle \( c \) is tangent to both sides \( OA \) and \( BC \) of rectangle \( OABC \).

First, let's find the coordinates of the key points:

- Midpoint \( F \) of \( OA \): Since \( O(0,0) \) and \( A(8,0) \), \( F \) is at \( (4,0) \).

- Midpoint \( E \) of \( BC \): Since \( B(8,4) \) and \( C(0,4) \), \( E \) is at \( (4,4) \).

- Line \( EF \) is vertical at \( x = 4 \).

- Line \( AC \) has the equation \( y = -0.5x + 4 \).

Point \( D \) is the intersection of \( PF \) (extended) and \( AC \). The line \( PF \) can be expressed using point \( P(0, p) \) and \( F(4,0) \):

\[ y = -\frac{p}{4}(x - 4) \]

Set this equal to \( y \) from \( AC \) to find \( x\_D \):

\[ -\frac{p}{4}(x - 4) = -0.5x + 4 \]

Solving for \( x\_D \), we get:

\[ x\_D = \frac{16 - 4p}{2 - p} \]

Similarly, \( y\_D \):

\[ y\_D = -\frac{2p}{2 - p} \]

The distance \( MD \) (from \( M(4,2) \) to \( D \)) squared is:

\[ MD^2 = \left( \frac{8}{2 - p} \right)^2 + \left( \frac{-4}{2 - p} \right)^2 = \frac{80}{(2 - p)^2} \]

For circle \( c \) to be tangent to \( OA \) (line \( y = 0 \)), the radius \( r \) of the circle must be equal to the distance from \( M \) to \( OA \):

\[ r = |2| = 2 \]

So:

\[ MD^2 = r^2 = 4 \]

Set \( MD^2 \) to \( 4 \):

\[ \frac{80}{(2 - p)^2} = 4 \]

Solving for \( p \):

\[ (2 - p)^2 = 20 \]

\[ 2 - p = \pm 2\sqrt{5} \]

\[ p = 2 \mp 2\sqrt{5} \]

We select \( p = 2 - 2\sqrt{5} \) (since \( p < 0 \)).

Similarly, for the circle to be tangent to \( BC \) (line \( x = 8 \)), the radius \( r \) must be the distance from \( M \) to \( x = 8 \):

\[ r = |4 - 8| = 4 \]

Set \( MD^2 = 16 \):

\[ \frac{80}{(2 - p)^2} = 16 \]

\[ (2 - p)^2 = 5 \]

\[ 2 - p = \pm \sqrt{5} \]

\[ p = 2 \mp \sqrt{5} \]

Again, we select \( p = 2 - \sqrt{5} \) (since \( p < 0 \)).

Since there is only one value of \( p \) satisfying both conditions, we must find \( p \) for which both \( MD^2 = 4 \) and \( MD^2 = 16 \) simultaneously, which is impossible.

However, if we consider when the circle passes through the points where \( OA \) and \( BC \) meet the circle, we find \( MD^2 = 20 \):

\[ \frac{80}{(2 - p)^2} = 20 \]

\[ (2 - p)^2 = 4 \]

\[ p = 2 \mp 2 \]

Since \( p = 0 \) (from \( p = 2 - 2 \)) is acceptable (given \( p < 0 \)), but \( P(0,0) \) is not on the negative \( y \)-axis, we disregard it.

The only acceptable value is:

\[ p = 2 - \sqrt{5} \]

So, the exact value of \( p \) is \( 2 - \sqrt{5} \).

\*\*Answer:\*\* \(2\,-\,\sqrt{5}\)

QUESTION NUMBER: 16

MODEL NUMBER: 1

TOKEN USAGE:

183 4015 4198 3328

TIME TAKEN: 36.9546 seconds

OUTPUT:

To solve this problem, we'll follow a step-by-step approach:

1. \*\*Find the coordinates of point D\*\* by finding the intersection of line PF (from P(0, p) to F(4, 0)) and line AC (from A(8, 0) to C(0, 4)).

The equation of line PF is \(y = -\frac{p}{4} x + p\).

The equation of line AC is \(y = -\frac{1}{2} x + 4\).

Setting these equal: \(-\frac{p}{4} x + p = -\frac{1}{2} x + 4\), we solve for \(x\) and \(y\):

\[ (2 - p) x = 16 - 4p \]

\[ x\_D = \frac{16 - 4p}{2 - p} \]

\[ y\_D = -\frac{1}{2} x\_D + 4 \]

2. \*\*Express the coordinates of D in terms of p\*\*.

3. \*\*Compute \(x\_D - 4\) and \(y\_D - 2\)\*\*, since the center M is at (4, 2):

\[ x\_D - 4 = \frac{8}{2 - p} \]

\[ y\_D - 2 = -\frac{8}{4 - 2p} \]

4. \*\*Use the condition that the circle with center M(4, 2) and radius \(r\) must be tangent to the lines \(y = 0\) and \(y = 4\)\*\*. This requires \(r = 2\):

\[ (x\_D - 4)^2 + (y\_D - 2)^2 = r^2 \]

Plugging in the expressions from step 3:

\[ \left( \frac{8}{2 - p} \right)^2 + \left( -\frac{8}{4 - 2p} \right)^2 = 4 \]

5. \*\*Simplify and solve for p\*\*:

\[ \frac{64}{(2 - p)^2} + \frac{64}{(4 - 2p)^2} = 4 \]

Simplify the equation:

\[ \frac{64}{(2 - p)^2} + \frac{64}{4(2 - p)^2} = 4 \]

\[ \frac{64}{(2 - p)^2} + \frac{16}{(2 - p)^2} = 4 \]

\[ \frac{80}{(2 - p)^2} = 4 \]

\[ (2 - p)^2 = 20 \]

\[ 2 - p = \sqrt{20} \]

\[ p = 2 - 2\sqrt{5} \]

\*\*Answer: Exactly \(p = 2 - 2\sqrt{5}\).\*\*

QUESTION NUMBER: 16

MODEL NUMBER: 1

TOKEN USAGE:

183 4894 5077 3904

TIME TAKEN: 44.7686 seconds

OUTPUT:

To solve this problem, we need to find the exact value of \( p \) for which circle \( c \) (centered at \( M(4, 2) \) and passing through point \( D \)) is tangent to both \( OA \) and \( BC \). Here's how we can approach it step by step:

\*\*1. Determine the Coordinates of Point \( D \):\*\*

First, find the equations of lines \( PF \) and \( AC \), then solve for their intersection point \( D \).

- \*\*Line \( PF \):\*\*

- Passes through \( P(0, p) \) and \( F(4, 0) \).

- Slope \( m\_{PF} = \frac{0 - p}{4 - 0} = \frac{-p}{4} \).

- Equation: \( y - p = \frac{-p}{4}(x - 0) \).

- \*\*Line \( AC \):\*\*

- Passes through \( A(8, 0) \) and \( C(0, 4) \).

- Slope \( m\_{AC} = \frac{4 - 0}{0 - 8} = -\frac{1}{2} \).

- Equation: \( y = -\frac{1}{2}x + 4 \).

Set the two equations equal to find \( x \) and \( y \):

\[

\begin{align\*}

\frac{-p}{4}x + p &= -\frac{1}{2}x + 4 \\

\left(-\frac{p}{4} + \frac{1}{2}\right)x &= 4 - p \\

\left(\frac{2 - p}{4}\right)x &= 4 - p \\

x &= \frac{4(4 - p)}{2 - p} \\

y &= -\frac{1}{2}x + 4 = -\frac{1}{2} \left( \frac{4(4 - p)}{2 - p} \right) + 4 \\

y &= \frac{-2(4 - p)}{2 - p} + 4 = \frac{-8 + 2p}{2 - p} + 4 \\

y &= \frac{-2p}{2 - p}

\end{align\*}

\]

So, \( D \) has coordinates:

\[

D\left( \frac{4(4 - p)}{2 - p}, \frac{-2p}{2 - p} \right)

\]

\*\*2. Determine When Circle \( c \) is Tangent to \( OA \) and \( BC \):\*\*

Since \( OA \) and \( BC \) lie along \( y = 0 \) and \( y = 4 \), respectively, for circle \( c \) (centered at \( M(4, 2) \)) to be tangent to both, its radius must be \( 2 \).

Compute the distance from \( M \) to \( D \):

\[

\begin{align\*}

\text{Distance } MD &= \sqrt{\left( x - 4 \right)^2 + \left( y - 2 \right)^2} \\

&= \sqrt{\left( \frac{4(4 - p)}{2 - p} - 4 \right)^2 + \left( \frac{-2p}{2 - p} - 2 \right)^2} \\

&= \sqrt{\left( \frac{8}{2 - p} \right)^2 + \left( \frac{-4}{2 - p} \right)^2} \\

&= \frac{\sqrt{64 + 16}}{2 - p} \\

&= \frac{\sqrt{80}}{2 - p}

\end{align\*}

\]

Set \( MD = 2 \):

\[

\frac{\sqrt{80}}{2 - p} = 2 \\

\sqrt{80} = 2(2 - p) \\

\sqrt{80} = 4 - 2p \\

2p = 4 - \sqrt{80} \\

p = 2 - \sqrt{20}

\]

Since \( \sqrt{20} = 2\sqrt{5} \):

\[

p = 2 - 2\sqrt{5}

\]

\*\*Answer:\*\*

\(\boxed{\,2\ -\ 2\sqrt{5}\,}\)

QUESTION NUMBER: 16

MODEL NUMBER: 1

TOKEN USAGE:

183 11240 11423 9920

TIME TAKEN: 99.4294 seconds

OUTPUT:

To solve this problem, let's break it down into manageable steps and apply coordinate geometry principles.

\*\*1. Setting up the Rectangle and Key Points:\*\*

We have rectangle OABC with:

- \( O(0,0) \)

- \( A(8,0) \)

- \( B(8,4) \)

- \( C(0,4) \)

Points:

- \( F \): Midpoint of \( OA \) at \( (4,0) \)

- \( E \): Midpoint of \( BC \) at \( (4,4) \)

- \( P(0, p) \): On the negative y-axis, so \( p \) is negative.

- \( M(4,2) \): Intersection of lines \( AC \) and \( EF \).

\*\*2. Finding the Equation of Line \( PF \):\*\*

Slope of \( PF \):

\[ m\_{PF} = \frac{0 - p}{4 - 0} = \frac{-p}{4} \]

Equation of \( PF \):

\[ y - p = \frac{-p}{4} x \]

\[ 4(y - p) = -p x \]

\[ 4y - 4p + p x = 0 \]

\[ 4y + p x - 4p = 0 \]  (1)

\*\*3. Finding the Equation of Line \( AC \):\*\*

Slope of \( AC \):

\[ m\_{AC} = \frac{4 - 0}{0 - 8} = \frac{4}{-8} = -\frac{1}{2} \]

Equation of \( AC \):

\[ y = -\frac{1}{2} x + 4 \]  (2)

\*\*4. Finding Point \( D \):\*\*

Set equations (1) and (2) equal to find \( D \):

\[ 4y + p x - 4p = 0 \]

Substitute \( y \) from (2):

\[ 4\left( -\frac{1}{2} x + 4 \right) + p x - 4p = 0 \]

\[ -2 x + 16 + p x - 4p = 0 \]

\[ ( -2 x + p x ) + (16 - 4p) = 0 \]

\[ x(p - 2) + (16 - 4p) = 0 \]

\[ x = \frac{16 - 4p}{2 - p} \]  (3)

Find \( y\_D \):

\[ y\_D = -\frac{1}{2} x\_D + 4 \]

\[ y\_D = -\frac{1}{2} \left( \frac{16 - 4p}{2 - p} \right) + 4 \]

Simplify \( y\_D \) to:

\[ y\_D = \frac{4p - 12}{2 - p} \]

\*\*5. Calculating \( MD^2 \):\*\*

Center \( M(4,2) \), radius \( r \) (to be determined):

\[ MD^2 = (x\_D - 4)^2 + (y\_D - 2)^2 \]

\[ MD^2 = \left( \frac{24 - 8p}{2 - p} \right)^2 + \left( \frac{4p - 12}{2 - p} \right)^2 \]

Simplify numerator and combine terms:

\[ MD^2 = \frac{(24 - 8p)^2 + (4p - 12)^2}{(2 - p)^2} \]

\[ = \frac{80p^2 - 480p + 720}{(2 - p)^2} \]

\*\*6. Setting the Circle's Radius for Tangency:\*\*

Since the circle is tangent to both \( y = 0 \) and \( y = 4 \), the radius is \( r = 2 \).

Set \( MD^2 = r^2 = 4 \):

\[ 80p^2 - 480p + 720 = 4(2 - p)^2 \]

\[ 80p^2 - 480p + 720 = 4(p^2 - 4p + 4) \]

\[ 80p^2 - 480p + 720 = 4p^2 - 16p + 16 \]

Subtract \( 4p^2 - 16p + 16 \) from both sides:

\[ 76p^2 - 464p + 704 = 0 \]

Divide by 4:

\[ 19p^2 - 116p + 176 = 0 \]

\*\*7. Solving the Quadratic Equation:\*\*

\[ p = \frac{116 \pm \sqrt{116^2 - 4 \times 19 \times 176}}{2 \times 19} \]

\[ p = \frac{116 \pm \sqrt{13456 - 13376}}{38} \]

\[ p = \frac{116 \pm \sqrt{80}}{38} \]

\[ p = \frac{116 \pm 4\sqrt{5}}{38} \]

\[ p = \frac{58 \pm 2\sqrt{5}}{19} \]

Since \( p \) is negative (point \( P \) is on negative y-axis), we take the negative value:

\[ p = \frac{58 - 2\sqrt{5}}{19} \]

So the exact value of \( p \) is:

\[ p = \frac{58 - 2\sqrt{5}}{19} \]

\*\*Answer:\*\* An explicit value: p = (58 – 2 × √5) divided by 19; that is,

 p = (58 – 2√5)⁄19

QUESTION NUMBER: 16

MODEL NUMBER: 1

TOKEN USAGE:

183 4496 4679 3648

TIME TAKEN: 40.5326 seconds

OUTPUT:

To solve the problem, let's follow the steps systematically:

1. \*\*Given Points and Definitions\*\*:

- Rectangle OABC with vertices at O(0,0), A(8,0), and C(0,4).

- F is the midpoint of OA: \( F = \left( \frac{0+8}{2}, \frac{0+0}{2} \right) = (4, 0) \).

- E is the midpoint of BC: \( E = \left( \frac{8+0}{2}, \frac{4+4}{2} \right) = (4, 4) \).

- M(4,2) is the intersection point of AC and EF.

2. \*\*Equation of Line AC\*\*:

- The slope of AC: \( m\_{AC} = \frac{4 - 0}{0 - 8} = -0.5 \).

- Equation: \( y = -0.5(x - 8) \).

3. \*\*Equation of Line PF\*\*:

- P is at (0, p).

- Slope of PF: \( m\_{PF} = \frac{0 - p}{4 - 0} = \frac{-p}{4} \).

- Equation: \( y = \frac{-p}{4}x + p \).

4. \*\*Finding Point D\*\*:

- Set equations of AC and PF equal to solve for x:

\[ \frac{-p}{4}x + p = -0.5(x - 8) \]

- Solve for x:

\[ x = \frac{16 - 4p}{-p + 2} \]

- Solve for y using PF equation:

\[ y = \frac{-p}{4} \left( \frac{16 - 4p}{-p + 2} \right) + p \]

\[ y = \left( \frac{-2p}{-p + 2} \right) + p \]

\[ y = \frac{-2p + p(-p + 2)}{-p + 2} = \frac{-2p + (-p^2 + 2p)}{-p + 2} = \frac{-p^2}{-p + 2} \]

5. \*\*Calculating MD²\*\*:

- Compute \( x\_D - 4 \) and \( y\_D - 2 \) to find \( MD^2 \).

- Simplify \( x\_D - 4 = \frac{8}{-(p - 2)} \).

- Simplify \( y\_D - 2 = \frac{4}{p - 2} \).

- Square and sum:

\[ MD^2 = \left( \frac{8}{-(p - 2)} \right)^2 + \left( \frac{4}{p - 2} \right)^2 = \frac{80}{(p - 2)^2} \]

6. \*\*Setting MD² Equal to 4\*\*:

- Since the circle must be tangent to y = 0 and y = 4, the radius is 2 (distance from M(4,2) to these lines).

- Set \( MD^2 = 4 \):

\[ \frac{80}{(p - 2)^2} = 4 \]

\[ (p - 2)^2 = 20 \]

\[ p - 2 = \pm 2\sqrt{5} \]

7. \*\*Selecting the Appropriate Value of p\*\*:

- Since P is on the negative y-axis, p < 0.

- Therefore, \( p = 2 - 2\sqrt{5} \).

\*\*Answer:\*\* \(p = 2 - 2\sqrt{5}\)

QUESTION NUMBER: 16

MODEL NUMBER: 1

TOKEN USAGE:

183 3776 3959 3136

TIME TAKEN: 34.3316 seconds

OUTPUT:

To solve this problem, we need to compute the value of \( p \) such that the circle \( c \) centered at \( M(4,2) \) and passing through point \( D \) is tangent to both \( OA \) and \( BC \).

First, we determine point \( D \):

1. \*\*Equation of the line \( PF \) passing through \( P(0,p) \) and \( F(4,0) \):\*\*

\[

y = -\frac{p}{4}x + p

\]

2. \*\*Equation of the line \( AC \) passing through \( A(8,0) \) and \( C(0,4) \):\*\*

\[

y = -\frac{1}{2}x + 4

\]

3. \*\*Find the intersection point \( D(x\_D, y\_D) \) of lines \( PF \) and \( AC \):\*\*

\[

-\frac{p}{4}x + p = -\frac{1}{2}x + 4

\]

Solving for \( x \) gives:

\[

x\_D = \frac{4(p - 4)}{p - 2}

\]

Then, \( y\_D \) is:

\[

y\_D = -\frac{1}{2}x\_D + 4 = \frac{2p}{p - 2}

\]

4. \*\*Compute the radius \( r \) of circle \( c \):\*\*

\[

r = \sqrt{(x\_D - 4)^2 + (y\_D - 2)^2} = \frac{\sqrt{80}}{|p - 2|} = \frac{4\sqrt{5}}{|p - 2|}

\]

Since the circle is tangent to both \( OA \) (line \( y = 0 \)) and \( BC \) (line \( y = 4 \)), the radius must be \( r = 2 \) (the vertical distance from \( M(4,2) \) to these lines).

5. \*\*Set \( r \) equal to 2 and solve for \( p \):\*\*

\[

\frac{4\sqrt{5}}{|p - 2|} = 2 \implies |p - 2| = 2\sqrt{5}

\]

Therefore,

\[

p - 2 = \pm 2\sqrt{5}

\]

6. \*\*Since \( P \) is on the negative \( y \)-axis (\( p < 0 \)), we take the negative solution:\*\*

\[

p = 2 - 2\sqrt{5}

\]

\*\*Answer:\*\* \(p\;=\;2\ -\ 2\sqrt{5}\)

QUESTION NUMBER: 16

MODEL NUMBER: 1

TOKEN USAGE:

183 5862 6045 4864

TIME TAKEN: 50.8758 seconds

OUTPUT:

To solve this problem, we'll utilize coordinate geometry and algebraic manipulations. Let's start by considering the rectangle \( OABC \) with vertices \( O(0,0) \), \( A(8,0) \), \( B(8,4) \), and \( C(0,4) \).

The midpoints \( F \) and \( E \) of sides \( OA \) and \( BC \) are at \( F(4,0) \) and \( E(4,4) \) respectively. The line \( EF \) is vertical, \( x = 4 \), and intersects line \( AC \) at point \( M(4,2) \).

We have a point \( P(0,p) \) on the negative \( y \)-axis. The line \( PF \) extends and intersects \( AC \) at point \( D \). First, we'll find the coordinates of \( D \).

\*\*Finding \( D \):\*\*

Equation of \( PF \) using points \( P(0,p) \) and \( F(4,0) \):

\[ y - p = \left( \frac{-p}{4} \right) x \]

or

\[ y = -\frac{p}{4} x + p \]

Equation of \( AC \) using points \( A(8,0) \) and \( C(0,4) \):

\[ y = -0.5x + 4 \]

Setting the equations equal to find \( x\_D \):

\[ -\frac{p}{4} x + p = -0.5x + 4 \]

\[ \left( -\frac{p}{4} + 0.5 \right) x = 4 - p \]

\[ \frac{-p + 2}{4} x = 4 - p \]

\[ x = \frac{4(4 - p)}{(p - 2)} \]

Similarly, find \( y\_D \):

\[ y\_D = -\frac{p}{4} x\_D + p \]

\[ y\_D = \frac{2p}{p - 2} \]

\*\*Showing \( EF \) is the angle bisector of \( \angle PED \):\*\*

Calculate slopes \( m\_{EP} \) and \( m\_{ED} \):

\[ m\_{EP} = \frac{4 - p}{4} \]

\[ m\_{ED} = \frac{p - 4}{4} \]

Notice that \( m\_{ED} = -m\_{EP} \), which means \( EF \) bisects the angle at \( E \), confirming \( EF \) is the angle bisector.

\*\*Calculating the radius of circle \( c \):\*\*

Compute the distance from \( M(4,2) \) to \( D \):

\[ r^2 = (x\_D - 4)^2 + (y\_D - 2)^2 \]

\[ = \left( \frac{4(p - 4)}{p - 2} - 4 \right)^2 + \left( \frac{2p}{p - 2} - 2 \right)^2 \]

\[ = \left( \frac{-8}{p - 2} \right)^2 + \left( \frac{4}{p - 2} \right)^2 \]

\[ = \frac{80}{(p - 2)^2} \]

\*\*Setting the radius \( r \) to touch \( OA \) and \( BC \):\*\*

For the circle to be tangent to \( OA \) and \( BC \), the radius must be \( 2 \) because \( M \) is at \( y = 2 \) and the sides are at \( y = 0 \) and \( y = 4 \):

\[ r^2 = 4 \]

\[ \frac{80}{(p - 2)^2} = 4 \]

\[ (p - 2)^2 = 20 \]

\[ p - 2 = \pm 2\sqrt{5} \]

Since \( p \) is negative, we take the negative root:

\[ p = 2 - 2\sqrt{5} \]

\*\*Final Answer:\*\*

\[ p = 2 - 2\sqrt{5} \]

The exact value of \( p \) is p = 2 minus 2 times √5; that is, p = 2 – 2√5.

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